Partial compensation/responsibility

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Abstract

Compensation/responsibility theory requires that differences in individual outcomes which can be fully attributed to differences in underlying 'compensation' factors should be eliminated, while differences in outcomes caused by differential 'responsibility' factors should be preserved. To implement the theory, a 'sharp' cut between compensation and responsibility factors has to be made, which is often difficult in practice. In this note, we introduce a more flexible 'soft' cut—based on a notion of partial compensation/responsibility—into a first-best income tax model à la Bossert (1995) and Bossert and Fleurbaey (1996). Two results emerge. First, we show that this 'soft' cut does not allow to escape the Bossert-Fleurbaey separability requirement of the gross income function. Second, we characterize a partial sharing rule-cum-separability as a natural candidate for partial redistribution.

1 Motivation

'Welfarism'—welfare in society is measured via an increasing function of subjective individual utilities—is the standard way in economics to assess, improve and optimize public policy. However, there are different reasons why using subjective utilities is objectionable. In "A Theory of Justice," Rawls (1971) criticizes the welfarist approach and argues in favour of equalizing an objective index of primary goods. In the aftermath of Rawls' work, many alternative theories of distributive justice were developed in Dworkin (1981a,b), Sen (1985), Arneson (1989), Cohen (1989), Roemer (1998), and Fleurbaey and Maniquet (2010). These new theories have the following selective-egalitarian viewpoint in common: equality is desirable, but only for differences in outcomes which are due to a selection of the underlying factors, the so-called 'compensation' factors; differences in outcomes due to the remaining 'responsibility' factors should be preserved.

To implement selective egalitarianism, a 'sharp' cut between compensation and responsibility factors has to be made.[1] Some factors, however, are neither pure compensation, nor pure responsibility factors. For example, education is a relevant factor for many outcomes, e.g., think of earnings, but it is usually considered to be influenced, among other things, by inborn talents (for which individuals cannot be held responsible), and exerted study effort (for which individuals are responsible). Unfortunately, these two underlying factors are not observed in practice and we are stuck with an observable factor, education, which cannot be unambiguously classified as either pure compensation or pure responsibility factor. In addition to the previous practical problem, opinion survey research shows that a 'soft' cut based on the

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[1] See Fleurbaey (2008) for useful references to applications.

idea of partial compensation/responsibility is closer to the opinions on distributive justice in different countries; see Schokkaert and Devooght (2003). These opinions could arise due to, e.g., a genuine belief in partial compensation/responsibility or because of second-best considerations.

In this note, we introduce partial compensation/responsibility in a first-best income tax framework à la Bossert (1995) and Bossert and Fleurbaey (1996).[2] Although income is the relevant outcome here, the model can also be applied to other outcomes like health expenditures (Schokkaert and Van de Voorde, 2004) and educational outcomes (Ooghe and Schokkaert, 2009). Gross income is modelled as a function of different factors which are partitioned into 'compensation' groups, i.e., subsets of factors with the same degree of compensation. The core axiom is partial compensation: differences in gross incomes which are only due to differences in factors belonging to one and the same compensation group should be *partially* reflected in differences in net incomes, depending on the degree of compensation for this group of factors. Two special cases arise. A compensation degree equal to zero implies no compensation and the existing gross income differences will be fully reflected in net income differences; A compensation degree equal to one leads to full compensation and results in equal net incomes.

We provide two main results. First, the introduction of a 'soft' cut based on the idea of partial compensation/responsibility does not allow to escape the Bossert-Fleurbaey separability result. More precisely, partial compensation requires the gross income function to be additively separable between the different compensation groups. Whether this is problematic or not is ultimately an empirical question. In case additive separability fits the data reasonably well, a simple partial sharing rule emerges as a natural candidate for partial redistribution. As a second result, we define and characterize this partial sharing rule-cum-additive separability based on three simple properties: budget balance (the sum of taxes must be equal to zero), equal treatment of equals (two individuals with the same type should receive the same net income), and partial solidarity (a multi-profile version of partial compensation).

2 Notation

Let $\mathbb{I}$ be a set of individuals with a cardinality denoted by I and let $\mathbb{J}$ be a set of factors with cardinality J. Each individual i in $\mathbb{I}$ is fully described by a type, i.e., a vector $x_i \in \mathbb{R}^J$. The gross income of an individual is a function of his type, formally, $g_i = G(x_i)$. The government wants to change the gross income distribution in society to obtain a more desirable net income scheme N. Such a net income scheme maps the type profile $\mathbf{x} = (x_i)_{i \in \mathbb{I}} \in \mathbb{D} \equiv \mathbb{R}^{I \times J}$ into a vector of net incomes $N(\mathbf{x})$; we use $n_i = N_i(\mathbf{x})$ to denote the net income of individual i. The difference between gross and net income is the tax (or subsidy, if negative), we use $t_i = g_i - n_i$, or, if confusion is possible, $T_i(\mathbf{x}) = G(x_i) - N_i(\mathbf{x})$.

Up to now, we follow the framework introduced in Bossert (1995) and Bossert and Fleurbaey (1996). Rather than partitioning the set of factors into either compensation or responsibility factors, we generalize the model here to allow for different groups of factors, each with a different degree of compensation. The social planner partitions the set of factors $\mathbb{J}$ into P different subsets denoted $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$ such that all factors with the same degree of compensation end up in the same 'compensation' group. We gather these degrees of compensation in a vector $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^P) \in \mathbb{R}^P$, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. As we will see

[2] In a companion paper, we introduce second-best considerations; see Ooghe and Peichl (2010).

later on, this includes, but is not restricted to the special cases of no compensation (full responsibility) if $\gamma^k = 0$ and full compensation (no responsibility) if $\gamma^k = 1$. For ease of exposition, we decompose the type of an individual as $x_i = \left(x_i^1, x_i^2, \ldots, x_i^P\right)$, with x_i^ℓ the subvector of x_i corresponding with the factors in compensation group ℓ in $\mathbb{P} \equiv \{1, 2, \ldots, P\}$. Note that the Bossert-Fleurbaey setting is a special case with two compensation groups ($P = 2$), one with full compensation and one with no compensation.

3 Partial compensation and additive separability

We define and link partial compensation and additive separability of the gross income function. We start with the core idea of partial compensation. Suppose that the gross income difference between two individuals can be fully attributed to differences in the factors of one compensation group only. In this case, partial compensation requires that the difference in taxes paid (or subsidies received) between these individuals should be proportional to their gross income difference, with the degree of compensation used as the proportionality factor. Equivalently, the net income difference between these individuals should be proportional to their gross income difference, now with the degree of responsibility—i.e., one minus the degree of compensation—as the proportionality factor.

Partial compensation. For each $\mathbf{x}$ in $\mathbb{D}$, for all individuals i, j in $\mathbb{I}$ and for each compensation group ℓ in $\mathbb{P}$, if $x_i^k = x_j^k$, for each k in $\mathbb{P} \setminus \{\ell\}$, then $t_i - t_j = \gamma^\ell (g_i - g_j)$, or equivalently, $n_i - n_j = \left(1 - \gamma^\ell\right)(g_i - g_j)$.

If the degree of compensation is zero for a compensation group, then full responsibility applies: both individuals have to pay the same taxes, or equivalently, the gross income difference is fully reflected in the net income difference. If the degree of compensation equals one, then full compensation applies: the tax difference reflects the gross income difference and as a result, both individuals receive the same net income. Besides these two special cases, a partial degree of compensation could apply to factors in compensation group k (if $0 < \gamma^k < 1$), but also undercompensation ($\gamma^k < 0$) and overcompensation ($\gamma^k > 1$) are possible.

Additive separability is the requirement that a function defined over types $x = \left(x^1, x^2, \ldots, x^P\right)$ in $\mathbb{R}^J$ (like the gross income function G) is additively separable in the different compensation groups.

Additive separability. Consider a partioning of $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$ and recall that a type $x \in \mathbb{R}^J$ can be decomposed as $x = \left(x^1, x^2, \ldots, x^P\right)$. A function $F : \mathbb{R}^J \to \mathbb{R} : x \mapsto F(x)$ is called additively separable (over the different compensation groups) if there exist functions $F^1, F^2, \ldots, F^P$ such that $F(x) = \sum_{k \in \mathbb{P}} F^k\left(x^k\right)$ for each $x = \left(x^1, x^2, \ldots, x^P\right) \in \mathbb{R}^J$.

Proposition 1 tells us that partial compensation requires the gross income function to be additively separable over the different compensation groups.[3]

Proposition 1. Let $I \geq 4$. If a net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ satisfies partial compensation then the gross income function G must satisfy additive separability.

Proof. See appendix. □

[3] It is also possible to define the axiom of partial compensation on the basis of income ratios, rather than income differences. A simple way to do this, is to replace all incomes and taxes in the current version of the axiom by its natural logarithm. This multiplicative version of partial compensation would lead to multiplicative separability of the gross income function.

4 A partial sharing rule

Whether the gross income function is additively separable, is ultimately an empirical question. The answer will depend on the data and on the chosen partitioning of the set of factors. If additive separability fits the data reasonably well, the following partial sharing rule is a natural candidate for partial redistribution.

Partial sharing rule. Consider a partioning of $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma = \left(\gamma^1, \gamma^2, \ldots, \gamma^P\right) \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. Suppose G is additively separable over the different compensation groups, i.e., there exist functions $G^1, G^2, \ldots, G^P$ such that $G(x)$ can be written as $\sum_{k\in\mathbb{P}} G^k\left(x^k\right)$ for each $x = \left(x^1, x^2, \ldots, x^P\right)$ in $\mathbb{R}^J$. The partial sharing rule assigns, for each $\mathbf{x}$ in $\mathbb{D}$, a net income

$$N_i(\mathbf{x}) \;=\; \underbrace{\frac{1}{I}\sum_{i\in\mathbb{I}}\sum_{k\in\mathbb{P}} \gamma^k G^k\left(x_i^k\right)}_{(1)} \;+\; \underbrace{\sum_{k\in\mathbb{P}}\left(1-\gamma^k\right) G^k\left(x_i^k\right)}_{(2)},$$

to individual i in $\mathbb{I}$.

The partial sharing rule equally shares those parts of individuals' gross incomes for which compensation applies in (1), and assigns the parts of gross income for which an individual is deemed responsible to that individual in (2). Note that a partial sharing rule looks like a 'basic income/differentiated flat tax'-proposal, with the shared part (1) as a basic income and the assigned part (2) calculated via differentiated linear tax rates (equal to the degrees of compensation).

It is easy to verify that the partial sharing rule also satisfies budget balance (the sum of gross incomes must be equal to the sum of net incomes) and equal treatment of equals (two individuals with the same type have to receive the same net income). Formally:[4]

Budget balance. For each $\mathbf{x}$ in $\mathbb{D}$, we have $\sum_{i\in\mathbb{I}} n_i = \sum_{i\in\mathbb{I}} g_i$.

Equal treatment. For each $\mathbf{x}$ in $\mathbb{D}$, for all i, j in $\mathbb{I}$, if $x_i = x_j$, then $n_i = n_j$.

To obtain a full characterization of the partial sharing rule, however, the previous three axioms (partial compensation, budget balance and equal treatment of equals) are not sufficient. One way to proceed is to replace partial compensation by partial solidarity. Partial solidarity is similar in spirit, but is concerned with changes in a profile $\mathbf{x}$. Suppose for example that the ℓ-th solidarity factor of individual j changes from x_j^ℓ to $x_j'^\ell$, ceteris paribus. Partial solidarity requires that the part of the resulting shock in the gross income of individual j for which (s)he is not reponsible should be borne equally by all individuals (including j), while the remaining part should be borne by individual j only.

partial solidarity. For all $\mathbf{x}, \mathbf{x}'$ in $\mathbb{D}$, for each j in $\mathbb{I}$ and for each ℓ in $\mathbb{P}$, if $x_i = x_i'$ for all i in $\mathbb{I}\setminus\{j\}$, and $x_j^k = x_j'^k$, for all k in $\mathbb{P}\setminus\{\ell\}$, then $N_j(\mathbf{x}') - N_j(\mathbf{x}) = N_i(\mathbf{x}') - N_i(\mathbf{x}) + \left(1-\gamma^\ell\right)\left(G\left(x_j'\right) - G\left(x_j\right)\right)$ for each i in $\mathbb{I}\setminus\{j\}$.

Before looking at the joint effect of the last three axioms together, we consider the effect of partial solidarity in combination with either budget balance or equal treatment of equals. First, partial solidarity

[4] It is possible to introduce an exogenous revenue requirement R in the budget constraint. This adds a constant term R/I to the partial sharing rule.

combined with budget balance clearly shows how a shock in the gross income of an individual is divided over the different individuals.

Lemma 1. Consider a partioning of $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma = \left(\gamma^1, \gamma^2, \ldots, \gamma^P\right) \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. Consider a net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ that satisfies partial solidarity and budget balance. Consider some $\mathbf{x}, \mathbf{x}'$ in $\mathbb{D}$, j in $\mathbb{I}$ and ℓ in $\mathbb{P}$ such that $x_i = x_i'$ for all i in $\mathbb{I}\backslash\{j\}$, and $x_j^k = x_j'^k$, for all k in $\mathbb{P}\backslash\{\ell\}$. We have

$$\begin{aligned} N_i(\mathbf{x}') - N_i(\mathbf{x}) &= \frac{\gamma^\ell}{I}\left(G\left(x_j'\right) - G\left(x_j\right)\right), \text{ for each } i \text{ in } \mathbb{I}\backslash\{j\}, \text{ and} \\ N_j(\mathbf{x}') - N_j(\mathbf{x}) &= \frac{\gamma^\ell}{I}\left(G\left(x_j'\right) - G\left(x_j\right)\right) + \left(1-\gamma^\ell\right)\left(G\left(x_j'\right) - G\left(x_j\right)\right). \end{aligned}$$

Proof. See appendix. □

Lemma 1 shows more clearly that a part γ^ℓ of the gross income shock $G\left(x_j'\right) - G\left(x_j\right)$ is shared equally, while the remaining part $\left(1-\gamma^\ell\right)\left(G\left(x_j'\right) - G\left(x_j\right)\right)$ is assigned to individual j.

Second, partial solidarity combined with equal treatment of equals implies partial compensation.

Lemma 2. Consider a partioning of $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma = \left(\gamma^1, \gamma^2, \ldots, \gamma^P\right) \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. If a net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ satisfies partial solidarity and equal treatment of equals, then it also satisfies partial compensation.

Proof. See appendix. □

Although partial solidarity is not stronger compared to partial compensation, lemma 2 tells us that it does provide somewhat more bite if it is combined with equal treatment of equals. This is also reflected in our final result, which provides a full characterization of the partial sharing rule-cum-additive separability.[5]

Proposition 2. Let $I \geq 4$. Consider a partioning of $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma = \left(\gamma^1, \gamma^2, \ldots, \gamma^P\right) \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. A net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ satisfies budget balance, equal treatment of equals, and partial solidarity if and only if

1. the gross income function G satisfies additive separability, *and*

2. the net income scheme N is the partial sharing rule.

Proof. See appendix. □

5 Conclusion

We have introduced a more flexible 'soft' cut—based on a notion of partial compensation/responsibility—into a first-best income tax model à la Bossert (1995) and Bossert and Fleurbaey (1996). This 'soft' cut does not allow to escape the Bossert-Fleurbaey separability requirement of the gross income function. If

[5] Proposition 2 also holds for $I \geq 2$, but, to make the proof short, it is based on proposition 1 (which requires $I \geq 4$).

additive separability fits the data reasonably well, we propose and characterize the partial sharing rule-cum-separability as a natural candidate for partial redistribution. From a theoretical point of view the two main results generalize some of the results in Bossert (1995) and Bossert and Fleurbaey (1996). But we also hope that the introduction of additional flexibility will make the compensation/responsibility theory more attractive for empirical implementation.

References

[1] Arneson, R., 1989, Equality and equal opportunity for welfare, *Philosophical Studies*, 56, 77-93.

[2] Bossert, W., 1995, Redistribution mechanisms based on individual characteristics, *Mathematical Social Sciences* 29, 1-17.

[3] Bossert, W., and Fleurbaey, M., 1996, Redistribution and compensation, *Social Choice and Welfare* 13, 343-355.

[4] Cohen, G.A., 1989, On the currency of egalitarian justice, *Ethics* 99/4, 906-944.

[5] Dworkin, R., 1981a, What is equality? Part I: equality of welfare, *Philosophy and Public Affairs*, 10, 185-246

[6] Dworkin, R., 1981b, What is equality? Part II: equality of resources, *Philosophy and Public Affairs*, 10, 283-345

[7] Fleurbaey, M., 2008, *Fairness, Responsibility and Welfare*, Oxford: Oxford University Press.

[8] Fleurbaey, M., Maniquet, F., 2010, Compensation and responsibility, chapter 22 in, Arrow, K.J., Sen, A.K. and Suzumura, K., eds., *Handbook of Social Choice and Welfare*, volume 2, North-Holland: Elsevier.

[9] Ooghe, E., Peichl, A., 2010, Optimal taxation and partial control: theory and evidence, mimeo.

[10] Ooghe, E., Schokkaert, E., 2009, School accountability: (how) can we reward schools and avoid cream-skimming? CORE discussion paper 09-22.

[11] Rawls, J., 1971, *A Theory of Justice*, Oxford: Oxford University Press.

[12] Roemer, J., 1998, *Equality of Opportunity*, Cambridge: Harvard University Press.

[13] Schokkaert, E., Devooght, K., 2003, Responsibility-sensitive fair compensation in different cultures, *Social Choice and Welfare* 21, 207-242.

[14] Schokkaert E., Van de Voorde, C., 2004, Risk selection and the specification of the conventional risk adjustment formula, *Journal of Health Economics* 23, 1237-1259.

[15] Sen, A., 1985, *Commodities and Capabilities*, Oxford: Oxford University Press.

Proof of proposition 1

Let $I \geq 4$. If a net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ satisfies PARTIAL COMPENSATION then we must show that the gross income function G satisfies ADDITIVE SEPARABILITY over the different compensation groups. Mor precisely, given the partioning of $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, there must exist functions $G^1, G^2, \ldots, G^P$, one function for each compensation group in $\mathbb{P} = \{1, 2, \ldots, P\}$, such that $G(x) = \sum_{k \in \mathbb{P}} G^k\left(x^k\right)$ for each $x = \left(x^1, x^2, \ldots, x^P\right) \in \mathbb{R}^J$. In case $J = 1$ or $P = 1$ the separability condition is obvious, so we focus on $J \geq 2$ and $2 \leq P \leq J$ in the sequel.

Step 1. Let J^k be the cardinality of $\mathbb{J}^k$. For any two compensation groups k and ℓ, with $\ell > k$, we show that there must exist functions $G_{k\ell}^{-\ell} : \mathbb{R}^{J - J_\ell} \to \mathbb{R}$ and $G_{k\ell}^{-k} : \mathbb{R}^{J - J_k} \to \mathbb{R}$ such that

$$G(x) = G_{k\ell}^{-\ell}\left(x^1, \ldots, x^{\ell-1}, x^{\ell+1}, \ldots, x^P\right) + G_{k\ell}^{-k}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^P\right)$$

for each $x \in \mathbb{R}^J$. Consider two compensation groups k and ℓ with $\ell > k$ and consider four individuals (1, 2, 3 and 4) with types

$$\begin{aligned}
x_1 &= \left(x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^{\ell-1}, x^\ell, x^{\ell+1}, \ldots, x^P\right) \equiv x, \\
x_2 &= \left(x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^{\ell-1}, b, x^{\ell+1}, \ldots, x^P\right), \\
x_3 &= \left(x^1, \ldots, x^{k-1}, a, x^{k+1}, \ldots, x^{\ell-1}, x^\ell, x^{\ell+1}, \ldots, x^P\right), \\
x_4 &= \left(x^1, \ldots, x^{k-1}, a, x^{k+1}, \ldots, x^{\ell-1}, b, x^{\ell+1}, \ldots, x^P\right).
\end{aligned}$$

for arbitrary vectors $x \in \mathbb{R}^J$, $a \in \mathbb{R}^{J^k}$ and $b \in \mathbb{R}^{J^\ell}$. Partial compensation requires

$$\begin{aligned}
n_1 - n_2 &= \left(1 - \gamma^\ell\right)(g_1 - g_2), && (1) \\
n_3 - n_4 &= \left(1 - \gamma^\ell\right)(g_3 - g_4), && (2) \\
n_1 - n_3 &= \left(1 - \gamma^k\right)(g_1 - g_3), && (3) \\
n_2 - n_4 &= \left(1 - \gamma^k\right)(g_2 - g_4). && (4)
\end{aligned}$$

Subtracting (2) from (1) and (4) from (3), and noting that both differences have to be the same, we get:

$$\left(1 - \gamma^\ell\right)(g_1 - g_2 - g_3 + g_4) = \left(1 - \gamma^k\right)(g_1 - g_3 - g_2 + g_4).$$

Given $\gamma_k \neq \gamma_\ell$ and $g_1 - g_2 - g_3 + g_4 = g_1 - g_3 - g_2 + g_4$, this is only possible if $g_1 - g_2 - g_3 + g_4 = 0$, or

$$G(x) = G(x_2) + (G(x_3) - G(x_4)),$$

for all vectors $x \in \mathbb{R}^J$, $a \in \mathbb{R}^{J^k}$ and $b \in \mathbb{R}^{J^\ell}$. Arbitrarily fixing a and b we can define

$$\begin{aligned}
G_{k\ell}^{-\ell}\left(x^1, \ldots, x^{\ell-1}, x^{\ell+1}, \ldots, x^P\right) &\equiv G(x_2) \\
&\equiv G\left(x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^{\ell-1}, b, x^{\ell+1}, \ldots, x^P\right)
\end{aligned}$$

and

$$\begin{aligned}
G_{k\ell}^{-k}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^P\right) &\equiv G(x_3) - G(x_4) \\
&\equiv G\left(x^1, \ldots, x^{k-1}, a, x^{k+1}, \ldots, x^{\ell-1}, x^\ell, x^{\ell+1}, \ldots, x^P\right) - \\
&\quad\; G\left(x^1, \ldots, x^{k-1}, a, x^{k+1}, \ldots, x^{\ell-1}, b, x^{\ell+1}, \ldots, x^P\right),
\end{aligned}$$

which leads to the desired result.

Step 2. On the basis of step 1, we show that there must exist a list of functions $G^1, G^2, \ldots, G^P$ s.t. $G(x) = \sum_{k\in\mathbb{P}} G^k(x^k)$ for each $x = (x^1, x^2, \ldots, x^P) \in \mathbb{R}^J$.

If $P = 2$, the representation follows directly from step 1. We proceed by induction. Consider P compensation groups, with $2 \le P < J$, and suppose that the existence of functions $G^{-\ell}_{k\ell} : \mathbb{R}^{J-J_\ell} \to \mathbb{R}$ and $G^{-k}_{k\ell} : \mathbb{R}^{J-J_k} \to \mathbb{R}$ for any two compensation groups k and ℓ, with $k < \ell \le P$, such that

$$G(x) = G^{-\ell}_{k\ell}\left(x^1, \ldots, x^{\ell-1}, x^{\ell+1}, \ldots, x^P\right) + G^{-k}_{k\ell}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^P\right) \tag{5}$$

holds for all $x \in \mathbb{R}^J$, implies additive separability of G (induction hypothesis). We show next that it also holds for $P+1$ groups. Consider a function with $P+1$ groups. From step 1 we know that, for each two compensation groups k and ℓ, with $k < \ell \le P+1$, there exist functions $G^{-\ell}_{k\ell} : \mathbb{R}^{J-J_\ell} \to \mathbb{R}$ and $G^{-k}_{k\ell} : \mathbb{R}^{J-J_k} \to \mathbb{R}$ such that

$$G(x) = G^{-\ell}_{k\ell}\left(x^1, \ldots, x^{\ell-1}, x^{\ell+1}, \ldots, x^{P+1}\right) + G^{-k}_{k\ell}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^{P+1}\right)$$

for each $x \in \mathbb{R}^J$. Using these conditions for arbitrary $k < \ell \le P$, and using the induction hypothesis, there must exist functions $\overline{G}^k\left(\cdot, x^{P+1}\right)$ for $k = 1, \ldots, P$, such that

$$G\left(x^1, \ldots, x^P, x^{P+1}\right) = \textstyle\sum_{k=1}^P \overline{G}^k\left(x^k, x^{P+1}\right), \tag{6}$$

for all $x^1, \ldots, x^P, x^{P+1}$. Now, consider an arbitrary compensation group $k < P+1$. Step 1 applied to k and $P+1$ gives us a representation

$$G\left(x^1, \ldots, x^P, x^{P+1}\right) = G^{-k}_{k(P+1)}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^{P+1}\right) + G^{-(P+1)}_{k(P+1)}\left(x^1, \ldots, x^P\right),$$

which can be combined with (6) to obtain

$$\textstyle\sum_{k=1}^P \overline{G}\left(x^k, x^{P+1}\right) = G^{-k}_{k(P+1)}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^{P+1}\right) + G^{-(P+1)}_{k(P+1)}\left(x^1, \ldots, x^P\right)$$

or equivalently,

$$\overline{G}\left(x^k, x^{P+1}\right) = G^{-k}_{k(P+1)}\left(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^{P+1}\right) + G^{-(P+1)}_{k(P+1)}\left(x^1, \ldots, x^P\right) - \textstyle\sum_{\ell\neq k} \overline{G}\left(x^\ell, x^{P+1}\right),$$

for all $x^1, \ldots, x^P, x^{P+1}$. Fixing all variables, except x^k and x^{P+1}, we get a representation of $\overline{G}\left(x^k, x^{P+1}\right)$ as

$$\overline{G}\left(x^k, x^{P+1}\right) = G^k\left(x^k\right) + \widetilde{G}^{P+1}_k\left(x^{P+1}\right),$$

with

$$\begin{aligned} G^k\left(x^k\right) &\equiv G^{-(P+1)}_{k(P+1)}\left(\overline{x}^1, ..., \overline{x}^{k-1}, x^k, \overline{x}^{k+1} \ldots, \overline{x}^P\right), \\ \widetilde{G}^{P+1}_k\left(x^{P+1}\right) &\equiv G^{-k}_{k(P+1)}\left(\overline{x}^1, \ldots, \overline{x}^{k-1}, \overline{x}^{k+1}, \ldots, \overline{x}^P, x^{P+1}\right) - \textstyle\sum_{\ell\neq k} \overline{G}\left(\overline{x}^\ell, x^{P+1}\right). \end{aligned}$$

Since this holds for any compensation group $k < P+1$ we can plug it in in equation (6) to obtain the desired result, i.e., the existence of functions G^k for $k = 1, \ldots, P+1$ such that

$$G\left(x^1, \ldots, x^P, x^{P+1}\right) = \textstyle\sum_{k=1}^{P+1} G^k\left(x^k\right),$$

for all $x^1, \ldots, x^P, x^{P+1}$, with $G^{P+1}\left(x^{P+1}\right)$ equal to $\sum_{k=1}^P \widetilde{G}^{P+1}_k\left(x^{P+1}\right)$.

Proof of lemma 1

Consider a partioning of all factors $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. Consider a net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ that satisfies PARTIAL SOLIDARITY and BUDGET BALANCE. Consider some $\mathbf{x}, \mathbf{x}'$ in $\mathbb{D}$, j in $\mathbb{I}$ and ℓ in $\mathbb{P}$ such that $x_i = x_i'$ for all i in $\mathbb{I} \backslash \{j\}$, and $x_j^k = x_j'^k$, for all k in $\mathbb{P} \backslash \{\ell\}$. Using partial solidarity, we must have

$$N_i(\mathbf{x}') - N_i(\mathbf{x}) = N_j(\mathbf{x}') - N_j(\mathbf{x}) - \left(1 - \gamma^\ell\right)\left(G\left(x_j'\right) - G(x_j)\right), \tag{7}$$

for each i in $\mathbb{I} \backslash \{j\}$. Summing both sides of equation (7) over i in $\mathbb{I} \backslash \{j\}$, we get

$$\textstyle\sum_{i \in \mathbb{I} \backslash \{j\}} \left(N_i(\mathbf{x}') - N_i(\mathbf{x})\right) = (I-1)\left(N_j(\mathbf{x}') - N_j(\mathbf{x}) - \left(1 - \gamma^\ell\right)\left(G\left(x_j'\right) - G(x_j)\right)\right). \tag{8}$$

Adding $N_j(\mathbf{x}') - N_j(\mathbf{x})$ to both sides of equation (8), we obtain

$$\textstyle\sum_{i \in \mathbb{I}} \left(N_i(\mathbf{x}') - N_i(\mathbf{x})\right) = I\left(N_j(\mathbf{x}') - N_j(\mathbf{x})\right) - (I-1)\left(1 - \gamma^\ell\right)\left(G\left(x_j'\right) - G(x_j)\right). \tag{9}$$

Using budget balance, we can rewrite the left hand side of equation (9) as

$$\textstyle\sum_{i \in \mathbb{I}} \left(N_i(\mathbf{x}') - N_i(\mathbf{x})\right) = \sum_{i \in \mathbb{I}} \left(G(x_i') - G(x_i)\right) = G\left(x_j'\right) - G(x_j). \tag{10}$$

Combining equation (9) and (10) leads to

$$N_j(\mathbf{x}') - N_j(\mathbf{x}) = \left(1 - \gamma^\ell\right)\left(G\left(x_j'\right) - G(x_j)\right) + \frac{\gamma^\ell}{I}\left(G\left(x_j'\right) - G(x_j)\right), \tag{11}$$

for individual j, as required. For the other individuals, plug in equation (11) in (7) to obtain

$$N_i(\mathbf{x}') - N_i(\mathbf{x}) = \frac{\gamma^\ell}{I}\left(G\left(x_j'\right) - G(x_j)\right),$$

for each i in $\mathbb{I} \backslash \{j\}$, which completes the proof.

Proof of lemma 2

Consider a partioning of all factors $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. Consider a net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ which satisfies PARTIAL SOLIDARITY and EQUAL TREATMENT OF EQUALS. We must show that also PARTIAL COMPENSATION is satisfied, more precisely, for each $\mathbf{x}$ in $\mathbb{D}$, for all individuals i, j in $\mathbb{I}$ and for each compensation group ℓ in $\mathbb{P}$ such that $x_i^k = x_j^k$, for each k in $\mathbb{P} \backslash \{\ell\}$ is true, $N_i(\mathbf{x}) - N_j(\mathbf{x}) = \left(1 - \gamma^\ell\right)\left(G(x_i) - G(x_j)\right)$ must result by combining partial solidarity and equal treatment of equals.

Construct a profile $\mathbf{x}'$ with (1) $x_l = x_l'$ for l in $\mathbb{I} \backslash \{j\}$, and (2) $x_j' = x_i$. In words, the transition from $\mathbf{x}$ to $\mathbf{x}'$ is such that individual j becomes a copy of individual i, ceteris paribus. This only requirs a change from x_j^ℓ to $x_j'^\ell = x_i^\ell$, while $x_j^k = x_j'^k$, for all k in $\mathbb{P} \backslash \{\ell\}$. Thus, we can apply partial solidarity, to get

$$N_j(\mathbf{x}') - N_j(\mathbf{x}) = N_i(\mathbf{x}') - N_i(\mathbf{x}) + \left(1 - \gamma^\ell\right)\left(G\left(x_j'\right) - G(x_j)\right), \tag{12}$$

for i and j. Now, since $x_i' = x_j'$ by construction, equal treatment of equals in profile $\mathbf{x}'$ requires $N_i(\mathbf{x}') = N_j(\mathbf{x}')$. Using $N_i(\mathbf{x}') = N_j(\mathbf{x}')$ and $x_j' = x_i$ in equation (12) leads to

$$N_i(\mathbf{x}) - N_j(\mathbf{x}) = \left(1 - \gamma^\ell\right)\left(G\left(x_j'\right) - G(x_j)\right) = \left(1 - \gamma^\ell\right)\left(G(x_i) - G(x_j)\right),$$

as required.

Proof of proposition 2

Let $I \geq 4$. Consider a partioning of all factors $\mathbb{J}$ into different compensation groups $\mathbb{J}^1, \mathbb{J}^2, \ldots, \mathbb{J}^P$, and let $\gamma = \left(\gamma^1, \gamma^2, \ldots, \gamma^P\right) \in \mathbb{R}^P$ collect the degrees of compensation for each compensation group, with $\gamma^k \neq \gamma^\ell$ if $k \neq \ell$. A net income scheme $N : \mathbb{D} \to \mathbb{R}^I : \mathbf{x} \mapsto N(\mathbf{x})$ satisfies BUDGET BALANCE, EQUAL TREATMENT OF EQUALS, and PARTIAL SOLIDARITY if and only if

1. the gross income function G satisfies ADDITIVE SEPARABILITY, and
2. the net income scheme N is a PARTIAL SHARING RULE.

It is easy to verify that, given additive separability, the partial sharing rule satisfies all axioms. We prove the opposite. In a first step, we show that additive separability of the gross income function G is implied by the axioms, while the second step provides us with the partial sharing rule.

Step 1. From lemma 2, we know that partial solidarity and equal treatment of equals imply partial compensation. Given $I \geq 4$, proposition 1 tells us that partial compensation implies additive separability, as required.

Step 2. From step 1 we know that there exist functions $G^1, G^2, \ldots, G^P$ s.t. $G\left(x^1, x^2, \ldots, x^P\right) = \sum_{k \in \mathbb{P}} G^k\left(x^k\right)$ for each $\left(x^1, x^2, \ldots, x^P\right)$ in $\mathbb{R}^J$. Let the set of individuals be $\mathbb{I} = \{1, 2, \ldots, I\}$. Consider an arbitrary profile $\mathbf{x}$ in $\mathbb{D}$ together with a sequence of profiles which converges to $\mathbf{x}$ as follows:

$$\begin{aligned}
{}_1\mathbf{x} &= \left({}_1x_1, {}_1x_2, \ldots, {}_1x_I\right) = \left(x_1, x_1, x_1, x_1, \ldots, x_1, x_1\right) \\
{}_2\mathbf{x} &= \left({}_2x_1, {}_2x_2, \ldots, {}_2x_I\right) = \left(x_1, x_2, x_1, x_1, \ldots, x_1, x_1\right) \\
{}_3\mathbf{x} &= \left({}_3x_1, {}_3x_2, \ldots, {}_3x_I\right) = \left(x_1, x_2, x_3, x_1, \ldots, x_1, x_1\right) \\
&\ldots \\
{}_I\mathbf{x} &= \left({}_Ix_1, {}_Ix_2, \ldots, {}_Ix_I\right) = \mathbf{x}.
\end{aligned}$$

Using equal treatment of equals, budget balance and step 1, we get

$$N_i\left({}_1\mathbf{x}\right) = G\left(x_1\right) = \textstyle\sum_{k \in \mathbb{P}} G^k\left(x_1^k\right), \tag{13}$$

for each i in $\mathbb{I}$. Focus on the change from profile ${}_1\mathbf{x}$ to ${}_2\mathbf{x}$. Using lemma 1—repeatedly, if necessary, since the type change for individual 2 might involve changes in more than 1 compensation group—, we get

$$\begin{aligned}
N_i\left({}_2\mathbf{x}\right) - N_i\left({}_1\mathbf{x}\right) &= \textstyle\sum_{k \in \mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right), \text{ for each } i \text{ in } \mathbb{I}\setminus\{2\} \\
N_2\left({}_2\mathbf{x}\right) - N_2\left({}_1\mathbf{x}\right) &= \textstyle\sum_{k \in \mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right) + \sum_{k \in \mathbb{P}}\left(1 - \gamma^k\right)\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right).
\end{aligned}$$

Given equation (13), we can rewrite these differences as

$$N_1\left({}_2\mathbf{x}\right) = \textstyle\sum_{k \in \mathbb{P}} G^k\left(x_1^k\right) + \sum_{k \in \mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right), \tag{14}$$

$$\begin{aligned}
N_2\left({}_2\mathbf{x}\right) &= \textstyle\sum_{k \in \mathbb{P}} G^k\left(x_1^k\right) + \sum_{k \in \mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right) \\
&\quad + \textstyle\sum_{k \in \mathbb{P}}\left(1 - \gamma^k\right)\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right),
\end{aligned} \tag{15}$$

$$N_i\left({}_2\mathbf{x}\right) = \textstyle\sum_{k \in \mathbb{P}} G^k\left(x_1^k\right) + \sum_{k \in \mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right), \text{ for each } i \text{ in } \mathbb{I}\setminus\{1, 2\}. \tag{16}$$

Focus now on the change from profile ${}_2\mathbf{x}$ to ${}_3\mathbf{x}$ and again using lemma 1, we get

$$\begin{aligned} N_i\left({}_3\mathbf{x}\right) - N_i\left({}_2\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_3^k\right) - G^k\left(x_1^k\right)\right), \text{ for each } i \text{ in } \mathbb{I}\backslash\left\{3\right\}, \\ N_3\left({}_3\mathbf{x}\right) - N_3\left({}_2\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_3^k\right) - G^k\left(x_1^k\right)\right) + \sum_{k\in\mathbb{P}}\left(1-\gamma^k\right)\left(G^k\left(x_3^k\right) - G^k\left(x_1^k\right)\right). \end{aligned}$$

This can be combined with equations (14)-(16) to get

$$\begin{aligned} N_1\left({}_3\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right) + \sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_3^k\right) - G^k\left(x_1^k\right)\right) \\ &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{j=1}^{3}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right) - 3\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_1^k\right)\right), \\ N_2\left({}_3\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{j=1}^{3}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right) - 3\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_1^k\right)\right) \\ &\quad \textstyle+ \sum_{k\in\mathbb{P}}\left(1-\gamma^k\right)\left(G^k\left(x_2^k\right) - G^k\left(x_1^k\right)\right), \\ N_3\left({}_3\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{j=1}^{3}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right) - 3\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_1^k\right)\right) \\ &\quad \textstyle+ \sum_{k\in\mathbb{P}}\left(1-\gamma^k\right)\left(G^k\left(x_3^k\right) - G^k\left(x_1^k\right)\right), \\ N_i\left({}_3\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{j=1}^{3}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right) - 3\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_1^k\right)\right), \text{ for each } i \text{ in } \mathbb{I}\backslash\left\{1,2,3\right\}, \end{aligned}$$

Proceeding in this way we end up at ${}_I\mathbf{x} = \mathbf{x}$ with

$$\begin{aligned} N_1\left({}_I\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{j=1}^{I}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right) - I\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_1^k\right)\right), \\ N_i\left({}_I\mathbf{x}\right) &= \textstyle\sum_{k\in\mathbb{P}} G^k\left(x_1^k\right) + \sum_{j=1}^{I}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right) - I\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_1^k\right)\right) \\ &\quad \textstyle+ \sum_{k\in\mathbb{P}}\left(1-\gamma^k\right)\left(G^k\left(x_i^k\right) - G^k\left(x_1^k\right)\right), \text{ for } i \text{ in } \mathbb{I}\backslash\left\{1\right\}. \end{aligned}$$

This can be rewritten to obtain

$$N_i\left({}_I\mathbf{x}\right) = N_i\left(\mathbf{x}\right) = \textstyle\sum_{k\in\mathbb{P}}\left(1-\gamma^k\right)G^k\left(x_i^k\right) + \sum_{j=1}^{I}\sum_{k\in\mathbb{P}} \frac{\gamma^k}{I}\left(G^k\left(x_j^k\right)\right), \text{ for each } i \text{ in } \mathbb{I},$$

which completes the proof.